OC576. Let x, y, z be real numbers such that the numbers

$$\frac{1}{|x^2+2yz|}, \qquad \frac{1}{|y^2+2zx|}, \qquad \frac{1}{|z^2+2xy|}$$

are side-lengths of a (non-degenerate) triangle. Find all possible values of the expression $xy+yz+zx$.

Originally Czech-Slovakia Math Olympiad, 2nd Problem, Category A, Final Round 2018.

We received 4 submissions, all of which were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

If $x = y = z = \alpha$, then

$$\left(\frac{1}{|x^2+2yz|}, \frac{1}{|y^2+2xz|}, \frac{1}{|z^2+2xy|}\right) = \left(\frac{1}{3\alpha^2}, \frac{1}{3\alpha^2}, \frac{1}{3\alpha^2}\right),$$

which are the side lengths of an equilateral triangle. In this case, $xy+yz+zx = 3\alpha^2$, so the expression takes on all positive values.

If $x = \alpha, y = -\alpha$, and $z = 0$, then

$$\left(\frac{1}{|x^2+2yz|}, \frac{1}{|y^2+2xz|}, \frac{1}{|z^2+2xy|}\right) = \left(\frac{1}{\alpha^2}, \frac{1}{\alpha^2}, \frac{1}{2\alpha^2}\right),$$

which again gives the side lengths of a triangle. Here, $xy+yz+zx = -\alpha^2$, so the expression also takes on all negative values.

Suppose that $xy+yz+zx = 0$. It is straightforward to verify that

$$\begin{aligned}\frac{1}{x^2+2yz} + \frac{1}{y^2+2xz} + \frac{1}{z^2+2xy} &= \frac{(xy+yz+zx)(2x^2+2y^2+2z^2+xy+yz+zx)}{(x^2+2yz)(y^2+2xz)(z^2+2xy)} \\ &= 0.\end{aligned}$$

After taking the absolute values of each term on the left, we must have one of

$$\frac{1}{|x^2+2yz|}, \frac{1}{|y^2+2xz|}, \frac{1}{|z^2+2xy|}$$

being a sum of the other two and the resulting triangle is degenerate.

In conclusion, $xy+yz+zx$ can assume any nonzero value.

OC577. Let $n \geq 2$ be an integer and let $A \in \mathcal{M}_n(\mathbb{C})$ such that A and A^2 have different ranks. Prove that there exists a nonzero matrix $B \in \mathcal{M}_n(\mathbb{C})$ such that $AB = BA = B^2 = O_n$.

Originally Romania Math Olympiad, 4th Problem, Grade 11, Final Round 2018.

Solution 2, by Missouri State University Problem Solving Group and UCLan Cyprus Problem Solving Group (done independently).

We know that A is similar to its Jordan canonical form, J, a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix} \text{ with } J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

That is, $P^{-1}AP = J$ for some invertible matrix P. It is easy to see that

$$\operatorname{rank}(A) = \operatorname{rank}(J) = \sum_{i=1}^{p} \operatorname{rank}(J_i).$$

Note that $\operatorname{rank}(J_i) = \operatorname{rank}(J_i^2)$ if $\lambda_i \neq 0$ because in that case J_i^2 is upper-triangular where each diagonal entry is λ_i^2. Therefore some eigenvalue must be zero. In fact, for A and A^2 to have different ranks, there must be some block, say J_p of the form

$$J_p = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix},$$

and of size $m \times m$ for some $m \geq 2$. Notice that $J_p^m = O_m$, but $J_p^{m-1} \neq O_m$. Define

$$B_1 = \begin{bmatrix} \mathbf{0} & & & \\ & \ddots & & \\ & & \mathbf{0} & \\ & & & J_p^{m-1} \end{bmatrix}.$$

Then $JB_1 = B_1 J = B_1^2 = O_n$. Since $A = PJP^{-1}$, if we set $B = PB_1P^{-1}$ then it follows easily that $AB = BA = B^2 = O_n$.